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The Stefan–Boltzmann constant in n-dimensional space

Peter T Landsberg† and Alexis De Vos‡

† Faculty of Mathematical Studies, University of Southampton, Southampton SO9 5NH, UK

‡ Laboratorium voor elektronika en meettechniek, Rijksuniversiteit te Gent, Sint Pietersnieuwstraat 41, B-9000 Gent, Belgium

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Abstract. For theoretical discussions of black-body radiation in a curved spacetime with n spatial dimensions, as envisaged in current superstring theories, one wants to know the n-dimensional generalisation of the main standard results. These are given in the present paper, but only for zero curvature, namely the Planck distribution, the Wien displacement law and the Stefan-Boltzmann law.

1. Introduction

The radiation energy flux Φ, emitted by a black body at temperature T, is given by the Stefan-Boltzmann law:

$$\Phi = \sigma A T^4 \equiv \int \phi_\nu \, d\nu \tag{1}$$

where A is the surface area of the radiating body and σ is the Stefan-Boltzmann constant:

$$\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2}$$

where k is the Boltzmann constant, h is Planck's constant and c is the speed of light. This result is influenced by the fact that space is three-dimensional. The present paper will derive the Stefan-Boltzmann law for an n-dimensional Euclidean space. We will see that the fourth power of the temperature in (1) will be replaced by an $(n+1)$th power:

$$\Phi = \sigma_n A T^{n+1}.$$

This property was already mentioned in a previous paper [1], but no explicit expression for the coefficient σ_n, i.e. the n-dimensional Stefan-Boltzmann constant, was calculated. We believe its value has not been published before. The present paper will present its derivation.

2. Calculation

Consider an n-dimensional cavity with the shape of a hypercube with length a_i in the x_i direction. Its volume V is thus $\Pi_i \, a_i$. Electromagnetic resonances consist of standing waves with a wavevector satisfying the boundary condition of ideally conducting walls:

$$k=\frac{1}{2}\left(\frac{m_1}{a_1},\frac{m_2}{a_2},\ldots,\frac{m_n}{a_n}\right)$$

where the m_i are positive integer numbers or zero.

The eigenvectors k thus generate an orthorhombic lattice in the n-dimensional k-space (figure 1). Using the de Broglie relation, the frequency

$$\nu=c|\boldsymbol{k}|$$

is associated with a particular eigenmode.

Therefore the number of modes g with frequency lower than ν is equal to the number of k lattice points in the hypersphere with radius ν/c. This sphere has a volume equal to

$$W_1=V_n(\nu/c)^n$$

where V_n denotes the volume of the n-dimensional hypersphere of radius 1. On the other hand, each lattice point occupies a unit cell with volume equal to

$$W_0=\frac{1}{2a_1}\frac{1}{2a_2}\cdots\frac{1}{2a_n}=\frac{1}{2^nV}.$$

If

$$W_1 \gg W_0 \tag{2}$$

i.e. if corner effects can be neglected, then the number of such cells in the hypersphere is

$$g=\frac{(1/2^n)W_1}{W_0}=V\frac{V_n}{c^n}\nu^n. \tag{3}$$

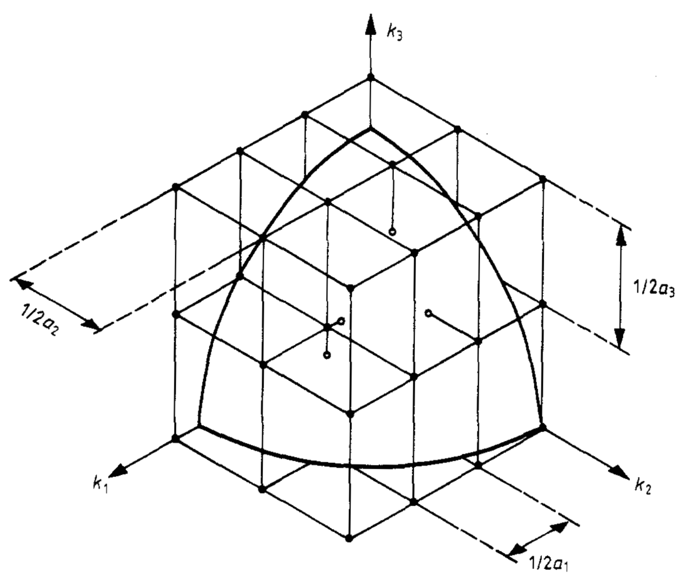

Figure 1. Orthorhombic lattice and sphere in k-space.

The volume W_1 has been divided by a factor 2^n because only the hyperoctant with exclusively positive coordinates is occupied by lattice points. The same result (3) can be obtained in a more fundamental way, without having to choose a particular cavity geometry (see appendix 1).

Finally we have to multiply the above result by a factor 2 because of the two independent polarisations of electromagnetic radiation. Hence we obtain, after differentiation, the following number of modes per unit frequency interval:

$$\frac{\mathrm{d}g}{\mathrm{d}\nu}=2V\frac{nV_n}{c^n}\nu^{n-1}.$$

Using the expression for the volume of the hypersphere (see, e.g., [2]), i.e.

$$V_n=\frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n+1)} \tag{4}$$

we obtain

$$\frac{\mathrm{d}g}{\mathrm{d}\nu}=\frac{4\pi^{n/2}}{\Gamma(n/2)}\frac{V}{c^n}\nu^{n-1} \tag{5}$$

where $\Gamma(x)$ stands for the gamma function.

Expressed as a function of the wavelength $\lambda=c/\nu$, this becomes

$$\frac{\mathrm{d}g}{\mathrm{d}\lambda}=\frac{4\pi^{n/2}V}{\Gamma(n/2)}\lambda^{-n-1}$$

a formula that has been mentioned previously (see problem (29.2) of [3]).

Having obtained the number of modes in a cavity, with frequency between ν and $\nu+\mathrm{d}\nu$, we now need to find the flux of such monochromatic photons, escaping the cavity through a radiating $(n-1)$-dimensional surface area A (figure 2).

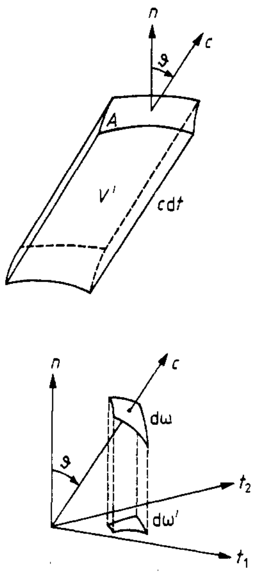

Figure 2. Photon flux through surface area A. The coordinate axis n is normal to the surface, while the axes $t_1, t_2, \ldots, t_{n-1}$ are tangent to it.

Since the radiation in the cavity is isotropic, i.e. uniformly distributed over all directions,

$$P=\frac{\mathrm{d}\omega}{\iint\ldots\int\mathrm{d}\omega}$$

is the probability of a particle to be directed in the elemental solid angle $\mathrm{d}\omega$. The number of photons with such a direction, hitting the surface area A in a time interval $\mathrm{d}t$, is the product of three factors:

(*a*) the number of modes per unit volume, i.e. expression (5) divided by V;

(*b*) the probability P; and

(*c*) the volume V' originally occupied by the photons, i.e. the hypercylinder volume $Ac\cos\vartheta\,\mathrm{d}t$.

Therefore we have to multiply (5) by

$$\frac{1}{V}Ac\cos\vartheta\,\mathrm{d}t\frac{\mathrm{d}\omega}{\iint\ldots\int\mathrm{d}\omega}.$$

After integration over all directions pointing out of the surface and after dividing by $\mathrm{d}t$, we obtain the multiplying factor

$$\iint\ldots\int\frac{Ac}{V}\cos\vartheta\frac{\mathrm{d}\omega}{\iint\ldots\int\mathrm{d}\omega}=\frac{Ac}{V}\frac{\iint\ldots\int\cos\vartheta\,\mathrm{d}\omega}{\iint\ldots\int\mathrm{d}\omega}.$$

In the latter formula both symbols $\iint\ldots\int$ stand for an $(n-1)$-dimensional integration, but the denominator is an integration over the n-dimensional unit *sphere*, whereas the numerator is an integration over an n-dimensional unit *hemisphere* (integration over all directions pointing out of the surface).

Now we remark that

$$\mathrm{d}\omega'=\cos\vartheta\,\mathrm{d}\omega$$

is the projection of $\mathrm{d}\omega$ onto the radiating hypersurface, so that the integral $\iint\ldots\int\cos\vartheta\,\mathrm{d}\omega$ over the n-dimensional hemihypersphere is equal to the volume of the $(n-1)$-dimensional unit hypersphere:

$$\iint\ldots\int\cos\vartheta\,\mathrm{d}\omega=V_{n-1}.$$

Note finally that

$$\iint\ldots\int\mathrm{d}\omega=S_n$$

where S_n denotes the surface area of the hypersphere with radius 1. The reader will easily verify that

$$S_n=nV_n \tag{6}$$

by remarking that the volume of a hypersphere equals the integration of its surface area:

$$V_nR^n=\int_0^R S_nr^{n-1}\,\mathrm{d}r.$$

Thus we have to multiply (5) by the factor

$$k_n = \frac{Ac}{V}\frac{V_{n-1}}{nV_n}$$

$$= \frac{Ac}{V}\frac{\Gamma(n/2)}{2\sqrt{\pi}\Gamma[(n+1)/2]}$$

which converts the numbers of modes in the cavity volume V into the numbers of modes piercing the cavity aperture A per unit time. Hence

$$k_1 = Ac/2V$$

$$k_2 = Ac/\pi V$$

$$k_3 = Ac/4V$$

$$k_4 = 2Ac/3\pi V$$

$$k_5 = 3Ac/16V$$

etc, and

$$\lim_{n\to\infty} k_n = Ac/(2\pi n)^{1/2} V.$$

For $n = 1$, the (zero-dimensional) surface area A is merely either one or two points. We have to replace A by 1 if only propagation in a single direction is considered or by 2 if propagation in both directions is considered. For $n = 3$, we find the familiar conversion factor $k = Ac/4V$, given in [4].

Applying the n-dimensional factor k_n to (5) yields

$$\frac{2A\pi^{(n-1)/2}}{\Gamma[(n+1)/2]}\frac{1}{c^{n-1}}\nu^{n-1}.$$

After multiplying by the photon energy $h\nu$ and by the Bose-Einstein factor $1/[\exp(h\nu/kT)-1]$, we obtain the n-dimensional Planck spectral density:

$$\phi_\nu = \frac{2A\pi^{(n-1)/2}}{\Gamma[(n+1)/2]}\frac{h}{c^{n-1}}\frac{\nu^n}{\exp(h\nu/kT)-1}. \tag{7}$$

Expressed in terms of the wavelength λ, the spectrum is

$$\phi_\lambda = \frac{2A\pi^{(n-1)/2}}{\Gamma[(n+1)/2]}hc^2\frac{\lambda^{-n-2}}{\exp(hc/\lambda kT)-1}.$$

Figure 3 shows some of these curves, normalised with respect to their maximum value. The maximum occurs at the wavelength λ obeying

$$\lambda = \frac{1}{x_n}\frac{hc}{kT} \tag{8}$$

with x_n the solution of the transcendental equation

$$[1 - x/(n+2)]\exp(x) = 1.$$

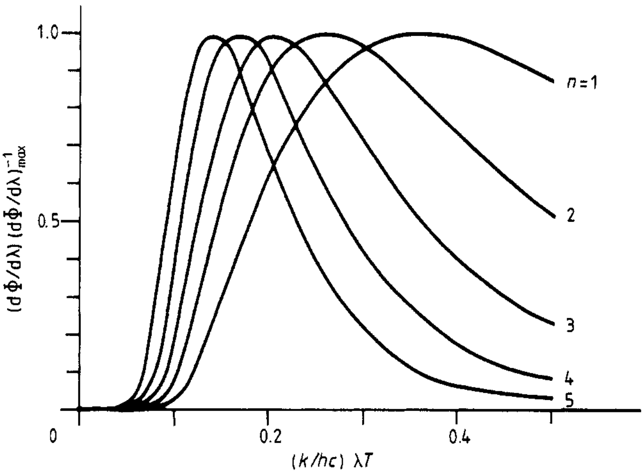

Figure 3. Normalised Planck spectrum as a function of the number of space dimensions n.

We have

$$x_1 = 2.8214$$

$$x_2 = 3.9207$$

$$x_3 = 4.9651$$

$$x_4 = 5.9849$$

$$x_5 = 6.9936$$

etc, and

$$\lim_{n\to\infty} x_n = n+2.$$

Formula (8) is the n-dimensional Wien displacement law.

After integrating over all frequencies ν, one obtains the n-dimensional version of the Stefan-Boltzmann law:

$$\Phi = \frac{2A\pi^{(n-1)/2}}{\Gamma[(n+1)/2]}\frac{h}{c^{n-1}}\int_0^\infty \frac{\nu^n\,d\nu}{\exp(h\nu/kT)-1}$$
$$= \frac{2A\pi^{(n-1)/2}}{\Gamma[(n+1)/2]}\Gamma(n+1)\zeta(n+1)\frac{k^{n+1}}{h^n c^{n-1}}\,T^{n+1} \tag{9}$$

or

$$\Phi = \sigma_n A T^{n+1}$$

with

$$\sigma_n = \frac{2\pi^{(n-1)/2}}{\Gamma[(n+1)/2]}\Gamma(n+1)\zeta(n+1)\frac{k^{n+1}}{h^n c^{n-1}}. \tag{10}$$

In the above formulae $\zeta(x)$ denotes the Riemann zeta function.

In particular we find

$$\sigma_1 = \frac{\pi^2 k^2}{3h} = 9.46 \times 10^{-13}\ \mathrm{W\,K^{-2}}$$

$$\sigma_2 = \frac{8\zeta(3)k^3}{h^2 c} = 1.92 \times 10^{-10}\ \mathrm{W\,m^{-1}\,K^{-3}}$$

$$\sigma_3 = \frac{2\pi^5 k^4}{15h^3 c^2} = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$$

$$\sigma_4 = \frac{64\pi\zeta(5)k^5}{h^4 c^3} = 2.01 \times 10^{-5}\ \mathrm{W\,m^{-3}\,K^{-5}}$$

$$\sigma_5 = \frac{8\pi^8 k^6}{63h^5 c^4} = 8.09 \times 10^{-3}\ \mathrm{W\,m^{-4}\,K^{-6}}$$

etc.

3. Discussion of the results

We immediately remark that $\sigma_3 = \sigma$, as it should. But also σ_1 is no novelty, as

$$\Phi = \frac{\pi^2 k^2}{3h} T^2$$

expresses the thermal noise power transfer in one-dimensional optical systems [5]. A more widely known case of one-dimensional thermal radiation is Johnson noise or Nyquist noise, i.e. noise in electrical networks, which however obeys [6]

$$\Phi = \frac{\pi^2 k^2}{6h} T^2.$$

This is caused by the fact that photons in an electrical network are polarised (with electric vector perpendicular to the metal conductors), so that an emissivity $\varepsilon = \frac{1}{2}$ has to be introduced for all the thermal sources, i.e. the electrical resistors. In other words, resistors are not black bodies, but grey bodies.

Whereas σ_1 and σ_3 are not new, we think that the other generalised Stefan-Boltzmann constants σ_n have not been published before. It is clear that σ_2 is applicable to integrated optics, where signals circulating in planar photonic devices, are corrupted with two-dimensional black-body noise.

Finally, the results for $n > 3$ can be useful in modern quantum field theory. It is indeed believed that a natural formulation of current superstring theories is in 10 or 11 dimensions for super-Yang-Mills and supergravity, respectively. For bosonic conformally symmetric string theories even 26 dimensions may be necessary, as reviewed, for example, in [7, 8]. In each of these theories, one of the dimensions represents time and three others represent ordinary space. The extra six, seven or 22 dimensions are spatial, but may register only if a very fine space scale is used, as they are in a sense 'crumpled up'. Nonetheless the situation may be different in the distant past or the far future of the expanding universe. (For relevant recent papers see, for example, [9-11].) Although our calculations are worked out for flat spaces, our results for $n = 9$,

$n=10$ and $n=25$ could possibly be of interest in connection with the above field theories:

$$k_9 = 35Ac/256V$$

$$k_{10} = 128Ac/315\pi V$$

$$k_{25} = 676\,039Ac/8388\,608V$$

and

$$\sigma_9 = \frac{32\pi^{14}k^{10}}{99h^9c^8}$$

$$\sigma_{10} = \frac{245\,760\pi^4\zeta(11)k^{11}}{h^{10}c^9}$$

$$\sigma_{25} = \frac{10\,779\,541\,504\pi^{38}k^{26}}{1403\,325h^{25}c^{24}}.$$

Note that $\zeta(11)$ in the above expression for σ_{10} can, without significant error, be replaced by unity. Indeed $\zeta(x)$ tends for $x \to +\infty$ exponentially to 1 and $\zeta(11)$ equals 1.000 49.

The generalisation from photons to any kind of extreme relativistic boson or fermion is given in appendix 2.

4. Application: a radiating hypersphere

As an example we now consider a hyperspherical radiation source, with radius R. We thus have the following radiating surface area:

$$A = S_n R^{n-1}. \tag{11}$$

Combining formulae (9) and (11), together with (4) and (6), yields

$$\Phi = \frac{4\pi^n \Gamma(n+1)\zeta(n+1)}{\sqrt{\pi}\Gamma(n/2)\Gamma[(n+1)/2]} \frac{k^{n+1}}{h^n c^{n-1}} R^{n-1}T^{n+1}.$$

Taking advantage of the following identity:

$$\Gamma(n) = \frac{2^{n-1}}{\sqrt{\pi}} \Gamma(n/2)\Gamma[(n+1)/2]$$

which is easily verified (and is known as the duplication formula of the gamma function [12]), one finally obtains

$$\Phi = \frac{2n\zeta(n+1)}{\pi}\left(\frac{2\pi k}{hc}\right)^n kcR^{n-1}T^{n+1}. \tag{12}$$

Figure 4 shows a logarithmic plot of the radiation flux Φ against temperature T. For convenience, both variables have been made dimensionless:

$$\Phi' = \frac{\Phi}{hc^2/\pi^2R^2}$$

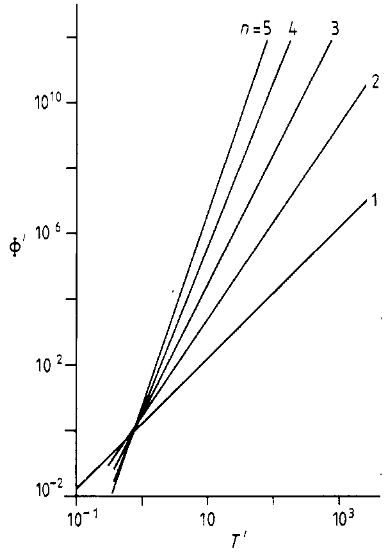

Figure 4. Black-body radiation intensity Φ as a function of temperature T (both in reduced units). The parameter n is the number of spatial dimensions.

and

$$T' = \frac{T}{hc/2\pi kR}.$$

We have therefore

$$\log(\Phi') = \log(n) + \log[\zeta(n+1)] + (n+1)\log(T').$$

All straight lines intersect in the vicinity of $T' = 1$. They are, however, not collinear! The reader will verify that, because of restriction (2), the radiation law (12) is valid only for T' sufficiently larger than 1. Therefore, we can say that, within its region of applicability, i.e. for a large enough RT product, the generalised Stefan-Boltzmann radiation intensity is an increasing function of the number of spatial dimensions.

5. Conclusion

Black-body radiation in n dimensions has here been analysed by standard methods and this has led to the generalised forms of the Planck energy spectrum, the Wien displacement law and the Stefan-Boltzmann law (equations (7), (8) and (9), respectively).

Apart from the intrinsic interest of the generalisations given in the present paper, its findings could also be relevant to the theories of electrical circuits, integrated optics and quantum fields.

We would like to conclude by giving a summary of the arguments used in § 2 in order to come to our final result (9), and of an alternative argument which might have been used, but was not.

First, the arguments applied were:

(i) compute the number of states (5) per unit frequency range;

(ii) obtain the energy flux (7) per unit frequency range by applying the factor k_n; and

(iii) obtain the energy flux and hence the Stefan-Boltzmann law by integrating over frequencies.

This procedure has the merit that, at stage (ii), one obtains the generalised laws of Planck and Wien.

Secondly, one might have proceeded from (i) in the following way.

(ii) Calculate the total energy content by integrating over frequencies, yielding (with a polarisation factor 2)

$$U = \frac{4\pi^{n/2}\Gamma(n+1)\zeta(n+1)}{\Gamma(n/2)} \frac{k^{n+1}T^{n+1}}{h^n c^n} V.$$

This formula occurs essentially in [3, equation (31.20)] and leads to the recognition of the system as an 'ideal quantum gas' in the sense of § 27 of [3], i.e. a system obeying $pV = gU$, with g a constant. In the present case g turns out to be $1/n$.

(iii) Obtain the total energy flux by applying the factor k_n to U.

The latter procedure gives U, but it misses the laws of Planck and Wien. The two procedures are, however, basically equivalent.

Acknowledgment

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Appendix 1

Suppose each mode occupies a volume h^n in phase space, according to the Heisenberg principle. Then the number of states in an infinitesimal element of phase space is

$$\mathrm{d}g = \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_n\, \mathrm{d}p_1\, \mathrm{d}p_2 \ldots \mathrm{d}p_n / h^n.$$

Integrating with respect to the n space coordinates over the volume V and with respect to the n momentum coordinates over a hypersphere with radius p yields

$$g = V V_n p^n / h^n$$

and thus (3), after substituting p by $h\nu/c$.

This alternative point of view is, in fact, the one followed by [1]. Comparison between the classical and the quantum approach is adequately discussed by Schrödinger [13] for the case $n = 3$.

Appendix 2

Whereas the thermodynamics of photons, as well as other relativistic bosons, takes advantage of the integral

$$\int_0^{+\infty} \frac{x^s\, \mathrm{d}x}{\mathrm{e}^x - 1} = \Gamma(s+1)\zeta(s+1) \tag{A2.1}$$

the statistics of extreme relativistic fermions (of negligible chemical potential) leads to the integral

$$\int_0^{+\infty} \frac{x^s \,\mathrm{d}x}{\mathrm{e}^x+1} = (1-2^{-s})\Gamma(s+1)\zeta(s+1) \tag{A2.2}$$

(see, e.g., [3]).

If P_b is the number of degrees of polarisation of a particular boson type and P_f the number of degrees of polarisation of a particular fermion type, we define the total number of boson and fermion types as

$$n_\mathrm{b} = \sum P_\mathrm{b}$$
$$\mathrm{n}_\mathrm{f} = \sum P_\mathrm{f}.$$

In expression (10) for the Stefan-Boltzmann constant we have to replace the photon polarisation factor 2 by n_b for the total boson contribution and by n_f for the fermion contribution. Finally, taking (A2.1) and (A2.2) into account, we get for the overall Stefan-Boltzmann constant:

$$\sigma_n = \frac{\pi^{(n-1)/2}}{\Gamma[(n+1)/2]}\Gamma(n+1)\zeta(n+1)\frac{k^{n+1}}{h^n c^{n-1}}[n_\mathrm{b}+(1-2^{-n})n_\mathrm{f}].$$

It should be noted that, for $n=9$, the result

$$\sigma_9 = \frac{16\pi^{14}}{99}\frac{k^{10}}{h^9 c^8}[n_\mathrm{b}+(1-2^{-9})n_\mathrm{f}]$$

is not in agreement with the value

$$\frac{8\pi^5}{3465}[n_\mathrm{b}+(1-2^{-9})n_\mathrm{f}]$$

mentioned in [9, 10]. We believe that, in the latter expression, the units are chosen such that $k=c=\hbar=1$, so that it corresponds to

$$\frac{8\pi^5}{3465}\frac{k^{10}}{\hbar^9 c^9}[n_\mathrm{b}+(1-2^{-9})n_\mathrm{f}] = \frac{4096\pi^{14}}{3465}\frac{k^{10}}{h^9 c^9}[n_\mathrm{b}+(1-2^{-9})n_\mathrm{f}] \tag{A2.3}$$

which (after multiplication by VT^{10}) is the correct value of the energy content in the volume V, but is not the energy flux through the surface A. In order to obtain the radiation flux one still has to multiply (A2.3) by $k_9 = 35Ac/256V$.

We believe that the authors of [9] make a misleading use of the term 'Stefan-Boltzmann constant', as the Stefan-Boltzmann law is related to energy fluxes through a surface and not to energy contents in a volume.

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